UNITED STATES
NAVAL POSTGRADUATE SCHOOL

A MODEL OF A SYSTEMS ANALYSIS STUDY

by

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ABSTRACT:

In this paper a mathematical model of an analysis teams results is presented. The variables are considered to have physical/social, time, space and state-of-nature attributes. The teams decision problem is formulated as a vector maximization problem.

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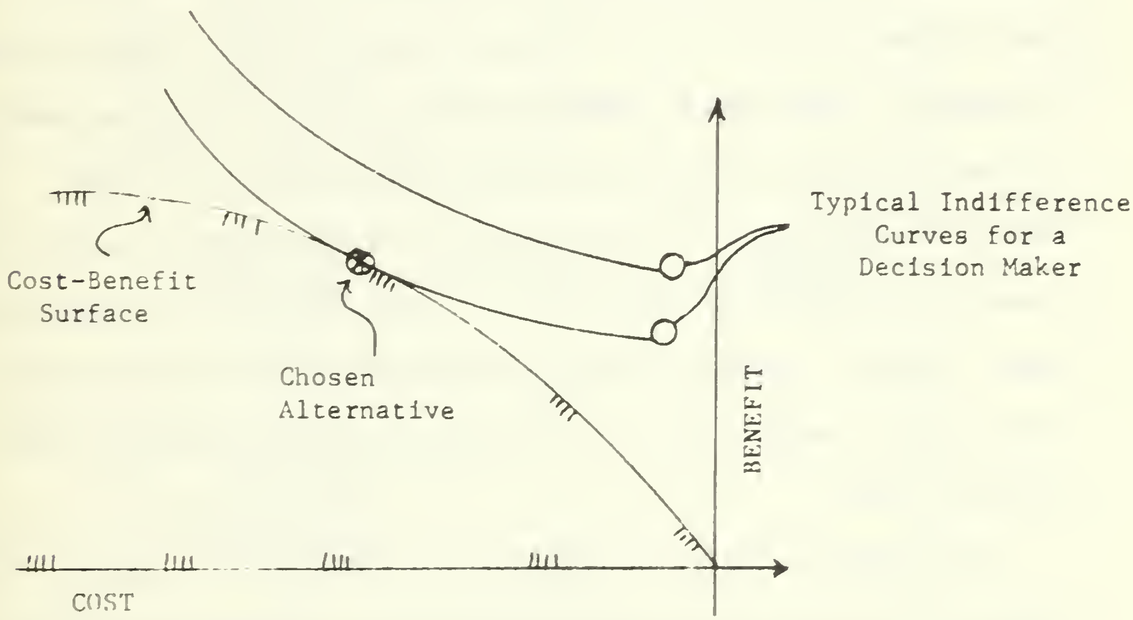
SECTION I

INTRODUCTION

This paper is concerned with a description of a governmental decision-maker choosing among alternatives whose costs and benefits have been illuminated analytically. The decision-maker is considered to be involved in a planning, programming and budgeting system and to be responsible for at least some area where cost-benefit studies can be helpful. The decision maker's study team is envisaged as being given an assignment to develop the alternatives and their costs and benefits. The output of the study team is some representation of a cost-benefit surface.



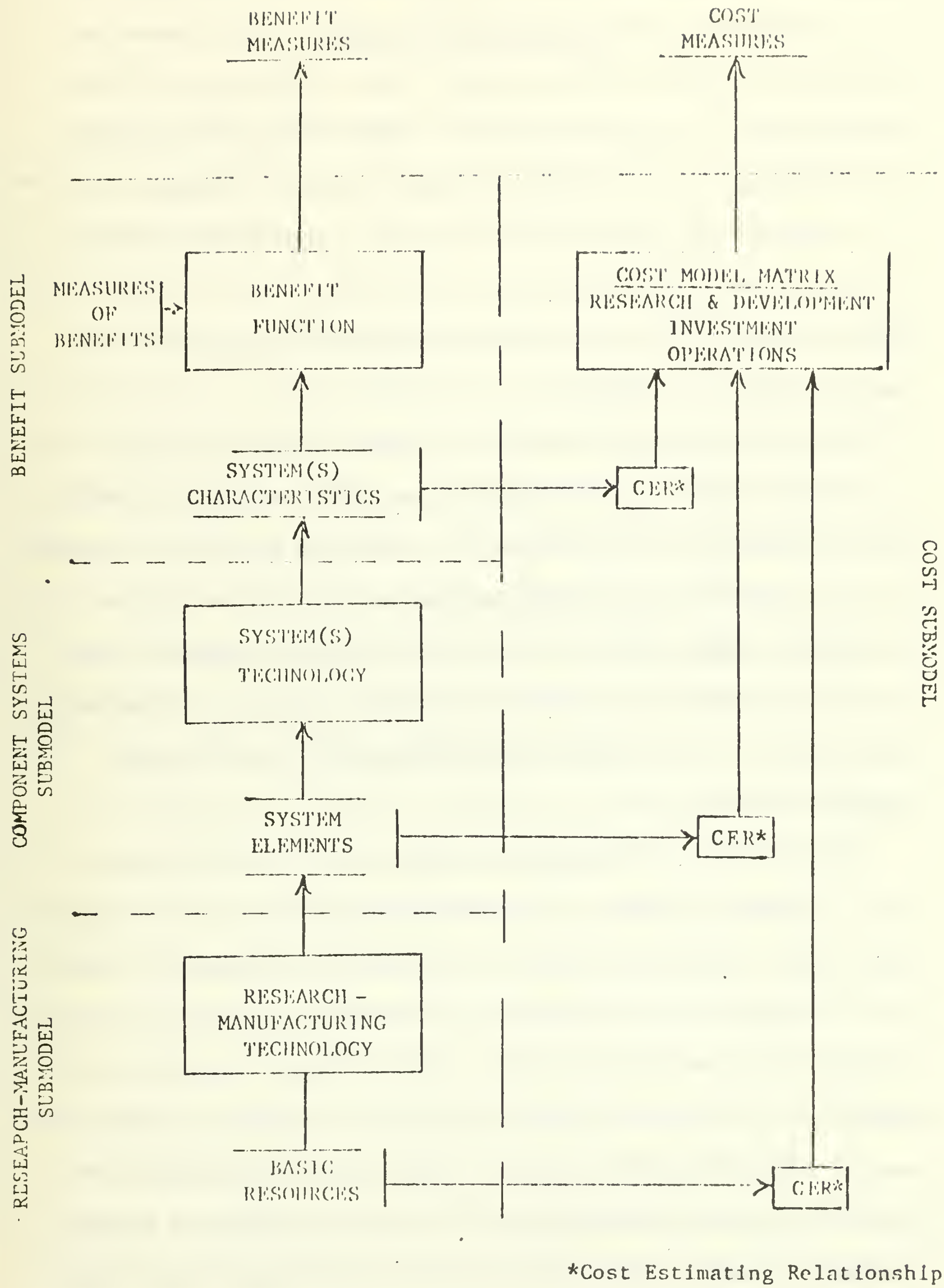
SIMPLIFIED GEOMETRY OF THE MODEL

FIGURE 1

The phenomenon to be described, then, is a governmental decisionmaker, his study team, and their interaction. In a very simplified fashion Figure 1 shows a cost-benefit surface developed by a study team. A decisionmaker's indifference curves are also shown, as is the subjectively selected optimum alternative. The details of the study team's deliberations are not shown by this figure. They are shown schematically in Figure 2.

As can be seen in the schematic, costs and benefits are produced by the study team as an interrelated flow among benefit, component systems, research-manufacturing, and cost submodels. The details of the operational definitions of the variables will be given in Section II. As can be seen by studying the schematic, basic resources (e.g., engineering hours, raw materials, tooling) are transformed into system elements (e.g., in the military context, tanks, planes, trained personnel). These system elements are the inputs to the component systems submodel. The outputs of this submodel are the system characteristics (e.g., in transportation, range, payload, speed, fuel consumption). These characteristics are produced from the system elements. Finally, characteristics are transformed into values of the system benefit measures (e.g., in poverty programs, expected income distributions).

The inputs to the cost model are characteristics, elements, and resources. By use of the cost estimating relationships, the cost model matrix can be computed and the cost measure(s) obtained. All these input types are considered to allow for such phenomenon as learning curves, quantity discounts, and rather detailed disaggregated



SCHEMATIC OF THE STUDY TEAM'S DELIBERATIONS

FIGURE 2

estimation. The cost model matrix has columns for the time periods of the analysis and rows for the system elements. The columns can be grouped by research and development costs, investment costs, and operating costs, if this is desirable. Some elements of the matrix may, of course, be zero. The cost measure values are computed by pre- and post-multiplication of the matrix by appropriate vectors. For example, if present costs are to be computed, then the pre-multiplication is by a sum vector and the post-multiplication is by a vector of discount factors.

Such a disaggregated model can be used to study the surface relating the various cost and benefit measures. Analogous to production theory in economics, this is a production possibility surface where each point is vectorially undominated. The surface can be considered as some function of all cost and benefit measures equal to zero - an implicit function. The implicit function is interpreted with benefits as outputs and costs as inputs and is the surface diagrammed in Figure 1.

The literature of mathematical models of such a phenomenon is small. Heuston and Ogawa [1] and the references there are the appropriate ones. This paper attempts to broaden the mathematical framework for describing the phenomenon of cost-benefit alternative choice. As such, it is somewhat a synthesis of the previous papers and also a generalization in that the previous models can be considered special cases of the model presented here. Another major difference is the stress here on the incommensurability of costs and benefits in many problems. As such, commensurability can be considered as net benefit measures, which, of course, are very desirable when available.

SECTION II

VARIABLE INTERPRETATION

In this section the variables in the model are given operational meaning. The variables are the benefit measures, the cost measures, the system characteristics, the system elements, and the basic resources.

As discussed in the previous section, the decisionmaker is modeled as choosing from a set of cost-benefit vectors that is generated by an analysis team. In this section the choice objects of the decisionmaker are given operational meaning. The choice objects are the benefit measures and the cost measures. These variables are assumed to have physical-social, time, space, and state-of-nature attributes. In addition to these variables, the exogenous variables where they are discussed in later sections also are assumed to have these attributes. The attributes will be discussed in turn.

The physical attributes of a measure have been discussed before [1]. It is stressed, though, that the same physical and/or social phenomenon can be measured in multiple ways - and they can all be important. For example, Miller, et al., [2] have listed the physical-social (my terminology) measures of poverty as income (threshold, relative, share of national income), assets (housing, consumer durables, savings, insurance), and services (education, health, neighborhood amenities, protection, social services, transportation). In considering this model, the reader is urged to regard some of the multiple measures as being associated with the same physical/social phenomenon.

The second attribute is time dating. With this attribute, the same physical/social measure at two different dates will be treated as two different measures. In this fashion, choice object time streams can be associated with a project. It is noted that the time attribute is associated with such measures as present cost and present benefits, since while they are calculated with many dates, they are calculated as of some particular date.

The third attribute locates the measure of the phenomenon in physical space. Hence, the same physical-social measure at two different locations will be treated as two different measures. A location is determined by categorizing the spatial extension of the phenomenon into elementary regions.

The risk or state-of-nature attribute will be modeled in the Debreusian manner [3]. That is, the future will be modeled as a time sequence of states-of-nature. At any one date, the states-of-nature are assertions concerning all that can conceivably happen including natural phenomenon, technological change, political acts, and the like. It is usual to model this as an event tree [4].

In cost-benefit analysis, particularly as used in the defense department, the scenario has been an important tool. A scenario seems to have no concise definition. However, it is used to mean the background aspects of a given situation. Here, scenario will be used to denote a unicursal path (a path with no steps retraced) through the event tree. Hence, in a model with only two dates (present and future), scenario and state-of-nature become synonymous. In summary, then, choice variables are defined to have an attribute for the state-of-nature that could prevail at a given date.

The above concept of state-of-nature is extended here to include the empirical relevance of alternative methods and models. As most practitioners have undoubtedly noticed during a study, discussion concerning the empirical relevance -- "realism"-- of alternative methods and models is often heated and lengthy. It is clear that such disagreement could be resolved by appropriate experimentation and application of scientific procedures. However, since the time frame of the decision does not always allow such experimentation and since the resources for such experimentation may not be available, an atrribute of empirical relevance is included in the concept of state-of-nature.

The choice objects, defined with physical-social, time, space, and risk attributes, must also be scaled and given mathematical structure. Here, the details of the scaling will not be considered [5]. Rather, each measure is assumed to have an associated multiplicative scale. This scale is represented by the real numbers.

SECTION III

THE COST-BENEFIT ANALYSIS

As discussed in the introduction, the analysis team is envisaged as developing the alternatives and the costs and benefits for these alternatives. In the disaggregated model considered here, the team begins with the research and development phase and studies the alternatives through the operational phase. The basic structure was given in Figure 2 in the introduction. The following discussion will consider each of the submodels, then consider the decision rules for efficiency. After completion of this topic, some attention will be given to comparative statics and an implicit function representation of the cost-benefit surface.

THE RESEARCH AND DEVELOPMENT-MANUFACTURING SUBMODEL

As shown in a schematic fashion in Figure 2, the inputs to this submodel are the basic resources and the outputs are the system elements. The basic resources will be designated by the letter x_k $(k = 1, \ldots, K)$, the system elements by y_j $(j = 1, \ldots, J)$. The technological transformation that represents the R&D-manufacturing process is assumed to be an implicit function involving the elements and resources. This implicit production function is written

$$G(\underline{y},\underline{x}) = 0 \; .$$

The bar beneath a variable designates a vector.

Various measures of technological trade-off are possible. Of interest here is (1) the trade-off between submodel outputs, (2) the trade-off between submodel inputs, and (3) the effect of an input on an output in the submodel. These trade-offs are shown in Table 1.

NAME OF TRADE-OFF	SYMBOL	FORMULA
RATE OF SYSTEM ELEMENT TRANSFORMATION	$RSETy_j y_\alpha$ $\alpha \neq j$	$\dfrac{\frac{\partial G}{\partial y_\alpha}}{\frac{\partial G}{\partial y_j}} = -\dfrac{\partial y_j}{\partial y_\alpha} = RSETy_j y_\alpha$
RATE OF BASIC RESOURCE SUBSTITUTION	$RBRSx_k x_\alpha$ $\alpha \neq k$	$\dfrac{\frac{\partial G}{\partial x_\alpha}}{\frac{\partial G}{\partial x_k}} = -\dfrac{\partial x_k}{\partial x_\alpha} = RBRSx_k x_\alpha$
MARGINAL PRODUCTIVITY OF RESOURCE k IN THE PRODUCTION OF ELEMENT j	$MPy_j x_k$	$-\dfrac{\frac{\partial G}{\partial x_k}}{\frac{\partial G}{\partial y_j}} = \dfrac{\partial y_j}{\partial x_k} = MPy_j x_k$

Technological Trade-Offs

Research and Development-Manufacturing Submodel

TABLE 1

The subscript α denotes either another output or input than the one subscripted by the j or k, respectively. This α notation will be used throughout the paper.

In many cases the submodel discussed here is not included in a study. Rather, either research and development and the details of manufacturing are not of interest, or R & D costs are estimated based on elements and/or characteristics. When either of these events occur, this submodel is not included and $\underline{x}$ does not appear.

THE COMPONENT SYSTEM SUBMODEL

As shown in a schematic manner in Figure 2, the inputs to this submodel are the system elements and the outputs are the system characteristics. The elements are designated as already discussed, while the letter z_i ($i = 1, \ldots, I$) will denote the i^{th} characteristic. The technology embodied in the component systems is represented by the implicit production function

$$F(\underline{z},\underline{y}) = 0 .$$

Table 2 charts the nature of the technological trade-offs applicable to the weapon system technology.

NAME OF TRADE-OFF	SYMBOL	FORMULA
RATE OF SYSTEM CHARACTERISTIC TRANSFORMATION	$RSCTz_i z_\alpha$ $\alpha \neq i$	$\dfrac{\partial F / \partial z_\alpha}{\partial F / \partial z_i} = -\dfrac{\partial z_i}{\partial z_\alpha} = RSCTz_i z_\alpha$
RATE OF SYSTEM ELEMENT SUBSTITUTION	$RSESy_j y_\alpha$ $\alpha \neq i$	$\dfrac{\partial F / \partial y_\alpha}{\partial F / \partial y_j} = -\dfrac{\partial y_i}{\partial y_\alpha} = RSESy_j y_\alpha$
MARGINAL COMPONENT SYSTEM PRODUCTIVITY OF ELEMENT j IN THE PRODUCTION OF CHARACTERISTIC i	$MCSPz_i y_j$	$-\dfrac{\partial F / \partial y_j}{\partial F / \partial z_i} = \dfrac{\partial z_i}{\partial y_j} = MCSPz_i y_j$

Technological Trade-Offs
The Component System Submodel

TABLE 2

THE BENEFIT SUBMODEL

As shown in a schematic manner in Figure 2, the inputs to this submodel are the system characteristics and the outputs the various measures of benefits. The characteristics are denoted as discussed and the various benefit measures by E_ℓ (ℓ = 1, ..., L). The technological relationships of the effectiveness submodel are represented by the implicit function

$$H(\underline{E},\underline{z}) = 0 .$$

Table 3 contains the information on the trade-offs applicable to the effectiveness submodel.

NAME OF TRADE-OFF	SYMBOL	FORMULA
RATE OF BENEFIT TRANSFORMATION	$RBTE_\ell E_\alpha$	$\dfrac{\frac{\partial H}{\partial E_\alpha}}{\frac{\partial H}{\partial E_\ell}} = -\dfrac{\partial E_\ell}{\partial E_\alpha} = RBTE_\ell E_\alpha$
RATE OF SYSTEM CHARACTERISTIC TRANSFORMATION	$RSCSz_i z_\alpha$	$\dfrac{\frac{\partial H}{\partial z_\alpha}}{\frac{\partial H}{\partial z_i}} = -\dfrac{\partial z_i}{\partial z_\alpha} = RSCSz_i z_\alpha$
MARGINAL BENEFIT OF THE i^{th} CHARACTERISTIC IN THE PRODUCTION OF THE ℓ^{th} BENEFIT MEASURE	$MBE_\ell z_i$	$-\dfrac{\frac{\partial H}{\partial z_i}}{\frac{\partial H}{\partial E_\ell}} = \dfrac{\partial E_\ell}{\partial z_i} = MBE_\ell z_i$

Technological Trade-Offs
The Benefit Submodel

TABLE 3

THE COST SUBMODEL

As shown schematically in Figure 2, the inputs to this submodel are basic resources $(x_k, k = 1, ..., K)$, system elements $(y_j, j = 1, ..., J)$, system characteristics $(z_i, i = 1, ..., I)$ and cost estimating parameters. The outputs are various cost measures. The measures are denoted by C_m $(m = 1, ..., M)$ and the cost estimating parameters by r_{mh} $(m = 1, ..., M; h = 1, ..., H)$. That is, r_{mn} is a cost estimating parameter and, in turn, is related to the statistical parameters in the individual cost estimating equations. The relationship between these variables is expressed as

$$C_m = C_m(\underline{z}, \underline{y}, \underline{x}, \underline{r}_m) \ .$$

Table 4 contains the interpretation of the various partial slopes of these cost measure functions.

Though the discussion in the remaining sections of this paper will be restricted to consideration of the above cost measure functions, some details will now be given to give the reader a better understanding of the functions. As discussed in the Introduction, the basic cost model for any measure type is a matrix with columns for time periods and rows for system elements. Since elements have an attribute of time, the cost model matrix, $\underline{\underline{C}}$, has nonzero elements only for the appropriate rows and columns. That is, $\underline{\underline{C}}$ can be partitioned into a diagonal matrix whose nonzero vectors are all on the diagonal. This is sketched in Figure 3. Some of the usual cost measures may be computed for this as shown below.

NAME OF PARTIAL SLOPE	SYMBOL	FORMULA
TOTAL MARGINAL m^{th} MEASURE COST OF THE i^{th} SYSTEM CHARACTERISTIC	$TMC_m z_i$	$\frac{\partial C_m}{\partial z_i} = TMC_m z_i$
TOTAL MARGINAL m^{th} MEASURE COST OF THE j^{th} SYSTEM ELEMENT	$TMC_m y_j$	$\frac{\partial C_m}{\partial y_j} = TMC_m y_j$
TOTAL MARGINAL m^{th} MEASURE COST OF THE k^{th} BASIC RESOURCE	$TMC_m x_k$	$\frac{\partial C_m}{\partial x_k} = TMC_m x_k$
MARGINAL m^{th} MEASURE COST DUE TO COST ESTIMATING PARAMETER r_{mh}	$MC_m r_{mh}$	$\frac{\partial C_m}{\partial r_{mh}} = MCC_m r_{mh}$

Cost Measure Partial Slopes

TABLE 4

$$\begin{bmatrix} \underline{c}^1 & \underline{0} & \underline{0} & \cdot & \cdot & \cdot & \underline{0} \\ \underline{0} & \underline{c}^2 & \underline{0} & \cdot & \cdot & \cdot & \underline{0} \\ \underline{0} & \underline{0} & \underline{c}^3 & \cdot & \cdot & \cdot & \underline{0} \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \underline{0} & \underline{0} & \underline{0} & \cdot & \cdot & \cdot & \underline{c}^T \end{bmatrix} = \underline{C}$$

The Cost Model Matrix
in Partitioned Form

FIGURE 3

where $\underline{c}^t$ = vector of cost estimating relationships applicable to period t.

Present Cost (PC)

The present cost measure is computed by multiplying each unit cost estimating relationship C_j by its corresponding element y_j ($j \in J^t$ where J^t denotes the indices for time period t). The formula for this is

$$\sum_{j \in J^t} y_j C_j = \underline{y}^T \underline{C} = \underline{O} .$$

The vector $\underline{O}$, a row vector, denotes the total outlays by time period. To complete the present cost calculation, the vector, $\underline{O}$, is multiplied by the vector of discount factors d_t ($t = 1, \ldots, T$). Thus,

$$PC = \underline{y}^T \underline{C} \underline{d} = \underline{O} \cdot \underline{d} = \sum_{t=1}^{T} \sum_{j \in J^t} d_t y_j C_j$$

Total Outlay (TO)

The computation here is the same as for present cost, except the discount factor vector $\underline{d}$ is now merely the sum vector $\underline{1}$.

0 Year System Cost (OC)

The 0 year system cost is similar to the total outlay. The difference is that only selected elements are used. The computation is performed by first multiplying a modified identity matrix by the element vector to get a column vector of selected elements. The identity matrix modification is the removal of the diagonal ones for those elements not selected. By formulae the computations are

$\underline{I}^{(S)} \underline{y}$	=	column vector of selected elements
$(\underline{I}^{(S)} \underline{y})^T \underline{C}$	=	row vector of yearly costs of selected elements
$(\underline{I}^{(S)} \underline{y})^T \underline{C}\, \underline{1}^{(S)}$	=	0 year system costs

The symbol $\underline{1}^{(S)}$ denotes the sum vector with zeros for years not of interest in the Θ year system costs.

Time Stream of Total Outlay

The measures of interest here are the outlays of costs in each time period. This will be a vector which is computed as $\underline{y}^T\underline{\underline{C}}$.

Time Stream of Selected Outlays

This measure is like the preceding except that only selected elements are used. The formula is $(\underline{\underline{I}}^{(S)}\underline{y})^T\underline{\underline{C}}$.

Unit Costs of System Elements

This measure is a vector measure of unit costs of each of the system elements. The formula is $\underline{\underline{C}}\underline{1}$.

In order to give a better understanding of the nature of the partial slopes of the cost measure functions, the next few paragraphs are concerned, first, with the individual elements of the cost model matrix and, second, the computation of appropriate partial slopes for some of the measures just discussed.

The individual CER is formulated as

$$C_j = f^{jt}(\underline{z},\underline{y},\underline{x},\underline{\gamma}_{jt}) \ .$$

The partial derivatives of this function with respect to the characteristics, elements, and resources measure the respective unit marginal costs. The partial derivative with respect to the cost coefficient γ_{hjt} measures the marginal j^{th} element cost with respect to its own k^{th} cost coefficient. When considering the marginal effect on system cost, the total marginal cost, all such unit marginal effects must be included. When the present cost measure (PC) is used, the computation is for the i^{th} characteristic.

$$TMC_{PC}z_i = \frac{\partial(\underline{y}^T \underline{C} d)}{\partial z_i} = \frac{\partial\left(\sum_{t=1}^{T} \sum_{j\varepsilon J} {}_t d_t y_j f^{jt}(\underline{z},\underline{y},\underline{x},\underline{y}_{jt})\right)}{\partial z_i}$$

$$= \sum_{t=1}^{T} \sum_{j\varepsilon J} {}_t d_t y_j \frac{\partial f^{jt}}{\partial z_i}$$

The result for total marginal cost is the weighted sum of the unit marginal costs with the weights being quantity of elements and the discount factor. When system elements are considered, the formula is

$$TMC_{PC}y_j = \frac{\partial(\underline{y}^t \underline{C} d)}{\partial y_j} = \sum_{t=1}^{T} \sum_{j\varepsilon J} {}_t d_t (f^{jt} + y_j \frac{\partial f^{jt}}{\partial y_j})$$

In this case there are two effects, since elements appear directly and as part of CER's. This same general pattern of weighted unit effects occurs for the other measures. This weighting is the reason for defining the partial slopes of the cost measure functions as total partial slopes. They are, in turn, usually complicated expressions.

ANALYSIS TEAM DECISION RULES

Using the submodels discussed in the previous paragraphs, the overall decision problem of the study team can be formulated. In this paper the team's decision problem will be formulated as a vector maximization problem. Background material on this type of formulation is given in References 6 and 7. The basic notion is to find that vector of benefit levels and cost levels such that there is no vector that will give more of one component without giving less of another.

In formal terms the analysis team's decision problem is

$$\text{"Max"} \begin{bmatrix} \underline{E} \\ --- \\ \underline{C} \end{bmatrix}$$

s.t.

$$H(\underline{E},\underline{z}) = 0$$

$$F(\underline{z},\underline{y}) = 0$$

$$G(\underline{y},\underline{x}) = 0$$

$$\underline{E},\underline{x},\underline{y},\underline{z} \geq 0$$

As the reader has undoubtedly observed, there are no restrictions on the signs of costs. This could be accomplished by adding additional constraints. This is not done here as no essential notion is lost by its exclusion. The usual assumptions concerning differentiability, constraint qualifications, and concavity/convexity are assumed. The Lagrangian of this problem is

$$\mathcal{L}(\underline{E},\underline{x},\underline{y},\underline{z},\underline{\lambda}) = \sum_{\ell=1}^{L} \phi_\ell E_\ell + \sum_{m=1}^{M} \psi_m C_m + \lambda_1 F(\underline{z},\underline{y}) +$$

$$\lambda_2 G(\underline{y},\underline{x}) + \lambda_3 H(\underline{E},\underline{z})$$

The necessary conditions for a maximum are as follows. A variable of the maximization problem, which appears below as a subscript, denotes a partial derivative with respect to that variable.

(1) $\phi_\ell + \lambda_3 H_{E_\ell} \leq 0 \qquad \ell = 1, \ldots, L$

$\hat{E}_\ell(\phi_\ell + \lambda_3 H_{E_\ell}) = 0 \qquad \hat{E}_\ell \geq 0$

(2) $\sum_{m=1}^{M} \psi_m C_{mz_i} + \lambda_1 F_{z_i} + \lambda_3 H_{z_i} \leq 0 \qquad i = 1, \ldots, I$

$\hat{z}_i\left(\sum_{m=1}^{M} \psi_m C_{mz_i} + \lambda_1 F_{z_i} + \lambda_3 H_{z_i}\right) = 0 \qquad \hat{z}_i \geq 0$

(3) $\sum_{m=1}^{M} \psi_m C_{my_j} + \lambda_1 F_{y_j} + \lambda_2 G_{y_j} \leq 0 \quad j = 1,\ldots,J$

$\hat{y}_j \left(\sum_{m=1}^{M} \psi_m C_{my_j} + \lambda_1 F_{y_j} + \lambda_2 G_{y_j} \right) = 0 \quad \hat{y}_j \geq 0$

(4) $\Sigma \psi_m C_{mx_k} + \lambda_2 G_{x_k} \leq 0 \quad k = 1,\ldots,K$

$\hat{x}_k (\Sigma \psi_m C_{mx_k} + \lambda_2 G_{x_k}) = 0 \quad \hat{x}_k \geq 0$

(5) $F(\underline{z},\underline{y}) = 0$

(6) $G(\underline{y},\underline{x}) = 0$

(7) $H(\underline{E},\underline{z}) = 0$

It is noted that the number of dependent variables $(\underline{E},\underline{x},\underline{y},\underline{z},\underline{\lambda})$ equals the number of equations.

The topic to be discussed next is the decision rules which can be derived from the necessary conditions. These decision rules are necessary for a maximum, and they are sufficient if the full concavity (convexity) assumptions are made. The rules presented below will be for the case where all variables are at a positive level and all the necessary conditions are equations. Further, the equations can be manipulated in various ways and only one possibility is presented here.

Decision Rule 1

Using equations 1, it is found that

$$\frac{\phi_\alpha}{\phi_\ell} = \frac{-\lambda_3 H_{E_\alpha}}{-\lambda_3 H_{E_\ell}} \Rightarrow \frac{\phi_\alpha}{\phi_\ell} = \frac{H_{E_\alpha}}{H_{E_\ell}} = RBTE_{\ell}E_{\alpha}.$$

This rule means that at an optimum the rate of transformation of benefit measures is equal to the appropriate ratio of ϕ's. In the standard economic literature the ϕ's are known as efficiency prices and they shall be so interpreted here. Since only relative efficiency prices are of interest as these measure the rates of benefit transformation, benefit E_ℓ is selected as numéraire.

Decision Rule 2

Using equations 1, 2, and the choice of numéraire, it is found that

$$\frac{\sum_{m=1}^{M} \left(\frac{\psi_m}{\phi_\ell}\right) (TMC_m z_\alpha) + (MBE_\ell z_\alpha)}{\sum_{m=1}^{M} \left(\frac{\psi_m}{\phi_\ell}\right) (TMC_m z_i) + (MBE_\ell z_i)} = \frac{F_{z_\alpha}}{F_{z_i}} = (RSCTz_i z_\alpha)$$

This rule is more easily interpreted if the C_m's are constrained to be negative numbers. The ψ's and ϕ's are strictly positive as shown by Karlin [Ref. 6, p. 217]. Then the first term in the numerator is the total variation in cost due to a change in the α^{th} characteristic. The sum of these terms, then, is the net change in units of benefit units of ℓ due to a direct effect on the benefit $(M E_\ell z_\alpha)$ and an indirect effect due to the cost measures. Overall, the rule says that the ratio of net variations in marginal benefit should equal the rate of characteristics transformation.

Decision Rule 3

Using equations 3 and the other decision rules, it is found that

$$\frac{\sum_{m=1}^{M} \frac{\psi_m}{\phi_\ell} C_{my_\alpha} + \left(\sum_{m=1}^{M} \frac{\psi_m}{\phi_\ell} C_{mz_i} + (MBE_\ell z_i)\right)(MCSPz_i y_\alpha)}{\sum_{m=1}^{M} \frac{\psi_m}{\phi_\ell} C_{my_j} + \left(\sum_{m=1}^{M} \frac{\psi_m}{\phi_\ell} C_{mz_i} + (MBE_\ell z_i)\right)(MCSPz_i y_j)} =$$

$$\frac{G_{y_\alpha}}{G_{y_j}} = RSETy_j y_\alpha$$

The first term in the numerator measures the marginal effect on all costs (in units of E_ℓ) of a change in y_α. The second term first measures the effect of y_α on z_i ($MCSPz_iy_\alpha$), then the effect of z_i on net units of E_ℓ. Again, there is an indirect effect of y_α on costs (first term) and a direct effect transformed to net units of E_ℓ. The overall numerator can be thought of as the net efficiency value of an additional unit of y_α measured in units of E_ℓ. The ratio of the net efficiency value of additional units of y_α and y_j must equal the rate of transformation of y_j and y_α at an optimum.

Decision Rule 4

Using equations 4, the decision rule is

$$\frac{\sum_{m=1}^{M} \frac{\psi_m}{\phi_\ell} C_{mx_\alpha}}{\sum_{m=1}^{M} \frac{\psi_m}{\phi_\ell} C_{mx_k}} = \frac{G_{x_\alpha}}{G_{x_k}} = MBRSx_kx_\alpha$$

The right side of this decision rule is the ratio of two total marginal costs. So the rule says to equate the rate of substitution to the ratio of total marginal costs. The costs are again measured in units of E_ℓ.

COMPARATIVE STATICS

The effect on the variables of the model of variation in the cost coefficients (γ's) is now discussed. This sensitivity analysis is performed in the usual manner considering the first order conditions of the problem discussed in the last section as implicitly defining a relationship between the variables and the γ's. The slopes of these functions are investigated. The result of the efforts in this section will be to show that the overall variation in the x's, y's, and z's can be considered as an efficiency substitution effect and a benefit effect. In accordance with the cost function notation, the letter r will denote the cost coefficient of interest.

Differentiating the equality necessary conditions of the last section yields the equations

$$
\begin{bmatrix}
\underline{A}_{11}^{L\times L} & \underline{A}_{12}^{L\times I} & \underline{0}^{L\times J} & \underline{0}^{L\times K} & \underline{0}^{L\times 1} & \underline{0}^{L\times 1} & \underline{A}_{17}^{L\times 1} \\
\underline{A}_{21}^{I\times L} & \underline{A}_{22}^{I\times I} & \underline{A}_{23}^{I\times J} & \underline{A}_{24}^{I\times K} & \underline{A}_{25}^{I\times 1} & \underline{A}^{I\times 1} & \underline{0}_{27}^{I\times 1} \\
\underline{0}^{J\times L} & \underline{A}_{32}^{J\times I} & \underline{A}_{33}^{J\times J} & \underline{A}_{34}^{J\times K} & \underline{A}_{35}^{J\times 1} & \underline{A}_{36}^{J\times 1} & \underline{0}^{J\times 1} \\
\underline{0}^{K\times L} & \underline{A}_{42}^{K\times I} & \underline{A}_{43}^{K\times J} & \underline{A}_{44}^{K\times K} & \underline{0}^{K\times 1} & \underline{A}_{46}^{K\times 1} & \underline{0}^{K\times 1} \\
\underline{0}^{1\times L} & \underline{A}_{25}^{1\times I} & \underline{A}_{35}^{1\times J} & \underline{0}^{1\times K} & 0 & 0 & 0 \\
\underline{0}^{1\times L} & \underline{0}^{1\times L} & \underline{A}_{36}^{1\times J} & \underline{A}_{46}^{1\times K} & 0 & 0 & 0 \\
\underline{A}_{17}^{1\times L} & \underline{A}_{27}^{1\times I} & \underline{0}^{1\times J} & \underline{0}^{1\times K} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \underline{z}}{\partial r} \\
\frac{\partial \underline{z}}{\partial r} \\
\frac{\partial \underline{y}}{\partial r} \\
\frac{\partial \underline{x}}{\partial r} \\
\frac{\partial \lambda_1}{\partial r} \\
\frac{\partial \lambda_2}{\partial r} \\
\frac{\partial \lambda_3}{\partial r}
\end{bmatrix}
=
\begin{bmatrix}
\underline{0}^{L\times 1} \\
\underline{b}_2^{I\times 1} \\
\underline{b}_3^{J\times 1} \\
\underline{b}_4^{K\times 1} \\
0 \\
0 \\
0
\end{bmatrix}
\qquad \text{(I)}
$$

The details of these quations can be seen in Table 5. For convenience of expression, set I is written as

$$\underline{A}\underline{q} = b.$$

The solution to this set of equations is then, in formal terms,

$$\underline{q} = \underline{A}^{-1}\underline{b}.$$

To understand this solution, it is necessary to consider the following efficiency problem. In this problem the idea is to vectorially maximize costs subject to a fixed level of effectiveness and the various technological transformations. In formal terms,

$$
\begin{array}{c}
 \\ (1) \\ (2) \\ (3) \\ (4) \\ (5) \\ (6) \\ (7)
\end{array}
\begin{array}{cccccccc}
E & z & y & x & \lambda_1 & \lambda_2 & \lambda_3 \\
\end{array}
$$

$$
\left[\begin{array}{ccccccc}
E & z & y & x & \lambda_1 & \lambda_2 & \lambda_3 \\
\left(\lambda_3 H_{E_\ell E_\ell}\right. & \left.\lambda_3 H_{E_\ell z_i}\right) & (0) & (0) & (0) & (0) & \left(H_{E_\ell}\right) \\
\left(\lambda_3 H_{z_i E_\ell}\right. & \left.\Sigma\psi_m C_{mz_i z_i} + \lambda_1 F_{z_i z_i} + \lambda_3 H_{z_i z_i}\right) & \left(\Sigma\psi_m C_{mz_i y_j} + \lambda_1 F_{z_i y_j}\right) & \left(\Sigma\psi_m C_{mz_i x_k}\right) & F_{z_i} & (0) & \left(H_{z_i}\right) \\
(0) & \left(\Sigma\psi_m C_{my_j z_i} + \lambda_1 F_{y_j z_i}\right) & \left(\Sigma\psi_m C_{my_j y_j} + \lambda_1 F_{y_j y_j} + \lambda_2 G_{y_j y_j}\right) & \left(\Sigma\psi_m C_{my_j x_k} + \lambda_2 G_{y_j x_k}\right) & \left(F_{y_j}\right) & \left(G_{y_j}\right) & (0) \\
(0) & \left(\Sigma\psi_m C_{mx_k z_i}\right) & \left(\Sigma\psi_m C_{mx_k y_j} + \lambda_2 G_{x_k y_j}\right) & \left(\Sigma\psi_m C_{mx_k x_k} + \lambda_2 G_{x_k x_k}\right) & (0) & \left(G_{x_k}\right) & (0) \\
(0) & \left(F_{z_i}\right) & \left(F_{y_j}\right) & (0) & (0) & (0) & (0) \\
(0) & (0) & \left(G_{y_j}\right) & \left(G_{x_k}\right) & (0) & (0) & (0) \\
\left(H_{E_\ell}\right) & \left(H_{z_i}\right) & (0) & (0) & (0) & (0) & (0)
\end{array}\right]
\left[\begin{array}{c}
\left(\frac{\partial E_\ell}{\partial \gamma}\right) \\
\left(\frac{\partial z_i}{\partial \gamma}\right) \\
\left(\frac{\partial y_j}{\partial \gamma}\right) \\
\left(\frac{\partial x_k}{\partial \gamma}\right) \\
\left(\frac{\partial \lambda_1}{\partial \gamma}\right) \\
\left(\frac{\partial \lambda_2}{\partial \gamma}\right) \\
\left(\frac{\partial \lambda_3}{\partial \gamma}\right)
\end{array}\right]
$$

$$
= \left((0), \left(-\Sigma\psi_m C_{mz_i\gamma}\right), \left(-\Sigma\psi_m C_{my_j\gamma}\right), \left(-\Sigma\psi_m C_{mx_k\gamma}\right), (0), (0), (0)\right)^T
$$

Equation Set I Details with Typical Elements

TABLE 5

$$\text{"Max"} \sum_{m=1}^{M} \psi_m C_m$$

$$\text{s.t.}$$
$$H(\underline{\mathring{E}},\underline{z}) = 0$$
$$F(\underline{z},\underline{y}) = 0$$
$$G(\underline{y},\underline{x}) = 0$$
$$\underline{x},\underline{y},\underline{z} \geq 0$$

where $\underline{\mathring{E}}$ designates the fixed level of all effectiveness measures. The notion of maximum is used as costs are treated as negative numbers. The Lagrangian for this problem is

$$\mathcal{L}^*(\underline{x},\underline{y},\underline{z},\underline{\mu}) = \sum_{m=1}^{M} \psi_m C_m + \mu_1 F(\underline{z},\underline{y}) + \mu_2 G(\underline{y},\underline{x}) + \mu_3 H(\underline{\mathring{E}},\underline{z}).$$

Again, the usual mathematical assumptions are made. The necessary conditions are

(A) $$\sum_{m=1}^{M} \psi_m C_{mz_i} + \mu_1 F_{z_i} + \mu_3 H_{z_i} \leq 0 \qquad i = 1,\ldots,I$$

$$\hat{z}_i\left(\sum_{m=1}^{M} \psi_m C_{mz_i} + \mu_1 F_{z_i} + \mu_3 H_{z_i}\right) = 0 \qquad \hat{z}_i \geq 0$$

(B) $$\sum_{m=1}^{M} \psi_m C_{my_j} + \mu_1 F_{y_j} + \mu_2 G_{y_j} \leq 0 \qquad j = 1,\ldots,J$$

$$\hat{y}_j\left(\sum_{m=1}^{M} \psi_m C_{my_j} + \mu_1 F_{y_j} + \mu_2 G_{y_j}\right) = 0 \qquad \hat{y}_j \geq 0$$

(C) $$\sum_{m=1}^{M} \psi_m C_{mx_k} + \mu_2 G_{x_k} \leq 0 \qquad k = 1,\ldots,K$$

$$\hat{x}_k\left(\sum_{m=1}^{M} \psi_m C_{mx_k} + \mu_2 G_{x_k}\right) = 0 \qquad \hat{x}_k \geq 0$$

(D) $F(\underline{z},\underline{y}) = 0$

(E) $G(\underline{y},\underline{x}) = 0$

(F) $H(\underline{\mathring{E}},\underline{z}) = 0$

The number of variables (x's, y's, z's, μ's) can be shown to equal the number of equations.

The relationships of these conditions to the original maximum problem are first studied by means of the Lagrange multipliers. Using equations (C) and (4), it can be shown that $\hat{\lambda}_2 = \hat{\mu}_2$ if the partial derivatives are evaluated at the same point. Equations (B) and (3) are used in conjunction with $\hat{\lambda}_2 = \hat{\mu}_2$ to obtain the theorem that $\hat{\mu}_1 = \hat{\lambda}_1$. Equations (A) and (2) in conjunction with $\hat{\mu}_1 = \hat{\lambda}_1$ are used to obtain $\hat{\lambda}_3 = \hat{\mu}_3$. Of course, all these equalities assume the partial derivatives are evaluated at the same point.

The second relationship between the efficiency problem and the maximum problem concerns the decision rules. With the relationship of the Lagrange multipliers it is clear that equations (A), (B), and (C) are the same as (2), (3), and (4). Hence, the decision rules are the same if derived only from these equations. The decision rules of the last section directly use (1), but this need not have been the case. It is concluded, then, that where applicable the two formulations led to the same decision rules.

To investigate the efficiency substitution effect, it is necessary to differentiate the necessary conditions with respect to the cost function parameter r. This procedure yields the equations

$$
\begin{bmatrix}
\underline{B}_{11}^{I\times I} & \underline{B}_{12}^{I\times J} & \underline{B}_{13}^{I\times K} & \underline{B}_{14}^{I\times 1} & \underline{0}^{I\times 1} & \underline{B}_{16}^{I\times 1} \\
\underline{B}_{21}^{J\times I} & \underline{B}_{22}^{J\times J} & \underline{B}_{23}^{J\times K} & \underline{B}_{24}^{J\times 1} & \underline{B}_{25}^{J\times 1} & \underline{0}^{J\times 1} \\
\underline{B}_{31}^{K\times I} & \underline{B}_{32}^{K\times J} & \underline{B}_{33}^{K\times K} & \underline{0}^{K\times 1} & B_{35}^{K\times 1} & \underline{0}^{K\times 1} \\
\underline{B}_{41}^{1\times I} & \underline{B}_{42}^{1\times J} & \underline{0}^{1\times K} & 0 & 0 & 0 \\
\underline{0}^{1\times I} & \underline{B}_{52}^{1\times J} & \underline{B}_{53}^{1\times K} & 0 & 0 & 0 \\
\underline{B}_{61}^{1\times I} & \underline{0}^{1\times J} & \underline{0}^{1\times K} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \underline{z}}{\partial r} \\
\frac{\partial \underline{y}}{\partial r} \\
\frac{\partial \underline{x}}{\partial r} \\
\frac{\partial \mu_1}{\partial r} \\
\frac{\partial \mu_2}{\partial r} \\
\frac{\partial \mu_3}{\partial r}
\end{bmatrix}
=
\begin{bmatrix}
\underline{b}_1^{I\times 1} \\
\underline{b}_2^{J\times 1} \\
\underline{b}_3^{K\times 1} \\
0 \\
0 \\
0
\end{bmatrix}
\qquad \text{(II)}
$$

In more compact notation:

$$\underline{\underline{B}}\,\underline{s} = \underline{t} \Rightarrow \hat{\underline{s}} = \underline{\underline{B}}^{-1} t$$

The details of this system of equations are shown in Table 6.

Inspection of I and II shows that II is, in fact, a submatrix of I. Using this information, I is written as the partitioned matrix:

$$
\left[\begin{array}{c|cccccc}
\underline{A}_{11}^{L\times L} & \underline{A}_{12}^{L\times I} & \underline{0}^{L\times J} & \underline{0}^{L\times K} & \underline{0}^{L\times 1} & \underline{0}^{L\times 1} & \underline{A}_{17}^{L\times 1} \\
\hline
\underline{A}_{21}^{I\times L} & & & & & & \\
\underline{0}^{J\times L} & & & & & & \\
\underline{0}^{K\times L} & & & \underline{\underline{B}}^{I+J+K+3\times I+J+K+3} & & & \\
\underline{0}^{1\times L} & & & & & & \\
\underline{0}^{1\times L} & & & & & & \\
\underline{A}_{71}^{1\times L} & & & & & &
\end{array}\right]
\left[\begin{array}{c}
\frac{\partial \underline{E}}{\partial r} \\
\hline
\frac{\partial \underline{z}}{\partial r} \\
\frac{\partial \underline{y}}{\partial r} \\
\frac{\partial \underline{x}}{\partial r} \\
\frac{\partial \lambda_1}{\partial r} \\
\frac{\partial \lambda_2}{\partial r} \\
\frac{\partial \lambda_3}{\partial r}
\end{array}\right]
=
\left[\begin{array}{c}
\underline{0}^{L\times 1} \\
\hline
\\
\\
\underline{t} \\
\\
\\
\\
\end{array}\right]
$$

$$
\begin{array}{c}
\begin{array}{c c c c c c c}
 & z & y & x & \mu_1 & \mu_2 & \mu_3 \\
(1) & \left(\Sigma\psi_m C_{mz_i z_i} + \mu_1 F_{z_i z_i} + \mu_3 F_{z_i z_i}\right) & \left(\Sigma\psi_m C_{mz_i y_j} + \mu_1 F_{z_i y_j}\right) & \left(\Sigma\psi_m C_{mz_i x_k}\right) & \left(F_{z_i}\right) & (0) & \left(H_{z_i}\right) \\
(2) & \left(\Sigma\psi_m C_{my_j z_i} + \mu_1 F_{y_j z_i}\right) & \left(\Sigma\psi_m C_{my_j} + \mu_1 F_{y_j y_j} + \mu_2 G_{y_j y_j}\right) & \left(\Sigma\psi_m C_{my_j x_k} + \mu_2 G_{y_j x_k}\right) & F_{y_j} & \left(G_{y_j}\right) & (0) \\
(3) & \left(\Sigma\psi_m C_{mx_k z_i}\right) & \left(\Sigma\psi_m C_{mx_k y_j} + \mu_2 G_{x_k y_j}\right) & \left(\Sigma\psi_m C_{mx_k x_k} + \mu_2 G_{x_k x_k}\right) & (0) & \left(G_{x_k}\right) & (0) \\
(4) & \left(F_{z_i}\right) & \left(F_{y_j}\right) & (0) & (0) & (0) & (0) \\
(5) & (0) & \left(G_{y_j}\right) & \left(G_{x_k}\right) & (0) & (0) & (0) \\
(6) & \left(H_{z_i}\right) & (0) & (0) & (0) & (0) & (0)
\end{array}
\begin{bmatrix} \frac{\partial z}{\partial \gamma} \\ \frac{\partial y}{\partial \gamma} \\ \frac{\partial x}{\partial \gamma} \\ \frac{\partial \mu_1}{\partial \gamma} \\ \frac{\partial \mu_2}{\partial \gamma} \\ \frac{\partial \mu_3}{\partial \gamma} \end{bmatrix}
=
\end{array}
$$

$$
\left(\left(-\Sigma\psi_m C_{mz_i}\right), \left(-\Sigma\psi_m C_{my_j}\right), \left(-\Sigma\psi_m C_{mx_k}\right), (0), (0), (0)\right)^T
$$

Equation Set II Details with Typical Elements

TABLE 6

Further, using the definitions,

$$\underline{Q}^{L\times I+J+K+3} = \begin{bmatrix} \underline{A}_{12} & \underline{0} & \underline{0} & \underline{0} & \underline{0} & \underline{A}_{17} \end{bmatrix}$$

$$\underline{P}^{I+J+K+3\times L} = \begin{bmatrix} \underline{A}_{21} \\ \underline{0} \\ \underline{0} \\ \underline{0} \\ \underline{0} \\ \underline{A}_{71} \end{bmatrix} \qquad \zeta = \begin{bmatrix} \dfrac{\partial \underline{z}}{\partial r} \\ \dfrac{\partial \underline{y}}{\partial r} \\ \dfrac{\partial \underline{x}}{\partial r} \\ \dfrac{\partial \lambda_1}{\partial r} \\ \dfrac{\partial \lambda_2}{\partial r} \\ \dfrac{\partial \lambda_3}{\partial r} \end{bmatrix}$$

the equations can be written

$$\begin{bmatrix} A_{11}^{L\times L} & \underline{Q}^{L\times I+J+K+3} \\ \underline{P}^{I+J+K+3\times L} & \underline{B}^{I+J+K+3\times I+J+K+3} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \underline{E}}{\partial r} \\ \zeta \end{bmatrix} = \begin{bmatrix} \underline{0}^{L\times 1} \\ \underline{t} \end{bmatrix}$$

The solution is:

$$\begin{bmatrix} \dfrac{\partial E}{\partial r} \\ \zeta \end{bmatrix} = \begin{bmatrix} \left((\underline{A}_{11}-\underline{Q}\,\underline{B}^{-1}\underline{P})^{-1}\right)^{L\times L} & \left(-(\underline{A}_{11}-\underline{Q}\,\underline{B}^{-1}\underline{P})^{-1}\underline{Q}\,\underline{B}^{-1}\right)^{L\times I+J+K+3} \\ \left(-\underline{B}^{-1}\underline{P}(\underline{A}_{11}-\underline{Q}\,\underline{B}^{-1}\underline{P})^{-1}\right)^{I+J+K+3\times L} & \left(\underline{B}^{-1}+\underline{B}^{-1}\underline{P}(\underline{A}_{11}-\underline{Q}\,\underline{B}^{-1}\underline{P})^{-1}\underline{Q}\,\underline{B}^{-1}\right)^{I+J+K+3\times I+J+K+3} \end{bmatrix} \begin{bmatrix} \underline{0}^{L\times 1} \\ \underline{t}^{I+J+K+3\times 1} \end{bmatrix}$$

so

$$\begin{bmatrix} \dfrac{\partial E}{\partial r} \\ \zeta \end{bmatrix} = \begin{bmatrix} -(\underline{A}_{11} - \underline{QB}^{-1}\underline{P})^{-1}\underline{QB}^{-1}\underline{t} \\ \underline{B}^{-1}\underline{t} + \underline{B}^{-1}\underline{P}(\underline{A}_{11} - \underline{QB}^{-1}\underline{P})^{-1}\underline{QB}^{-1}\underline{t} \end{bmatrix}$$

Notice that the solution for $\frac{\partial E}{\partial r}$ also appears in the solution for $\underline{\zeta}$. Also $\underline{\hat{s}}$ appears. Explicitly then, the solution for $\underline{\zeta}$ is:

$$\underline{\zeta} = \underline{\hat{s}} - \underline{B}^{-1}\underline{P}\,\frac{\partial E}{\partial r}$$

The first term of this equation is the efficiency substitution effect, and the second term is the benefit effect. Thus, a change in a cost function parameter can be considered to have two additive components. The first component is the variation in the $\underline{z}$, $\underline{x}$, or $\underline{y}$ due to the variation in the cost coefficient holding the effectiveness level constant. The second component is the effect on the $\underline{z}$, $\underline{x}$, or $\underline{y}$ due to the effect on benefit due to the cost coefficient. This latter component effect is due to variations in the $\underline{z}$, $\underline{x}$, $\underline{y}$ in the technologies and cost functions.

In a manner analogous to traditional economic theory, substitutes and complements can be defined. For efficiency substitutes and complements the definitions are:

<u>Efficiency Substitutes</u>

Two characteristics (elements, resources) are called efficiency substitutes for cost function parameter r_{mh} if

$$\left.\frac{\partial z_i}{\partial r_{mh}}\right|_{\underline{E}} > 0 \quad \text{and} \quad \left.\frac{\partial z_\alpha}{\partial r_{mh}}\right|_{\underline{E}} < 0 \qquad i \neq \alpha$$

<u>Efficiency Complements</u>

Two characteristics (elements, resources) are called efficiency complements for cost function parameter r_{mh} if

$$\left.\frac{\partial z_i}{\partial r_{mh}}\right|_{\underline{E}} > 0 \quad \text{and} \quad \left.\frac{\partial z_\alpha}{\partial r_{mh}}\right|_{\underline{E}} > 0 \qquad i \neq \alpha$$

The next comparative statics problem to consider is the one associated with the efficiency prices. This problem is of interest since it is the variation in these prices which "sweeps out" the cost-benefit surface.

Again, the necessary conditions for the vector maximum are differentiated with respect to the variable of interest, which is now the efficiency price ψ_m associated with the m^{th} cost measure. The following equations are obtained.

$$\underline{A} \begin{bmatrix} \frac{\partial \underline{E}}{\partial \psi_m} \\ \frac{\partial \underline{E}}{\partial \psi_m} \\ \frac{\partial \underline{y}}{\partial \psi_m} \\ \frac{\partial \underline{x}}{\partial \psi_m} \\ \frac{\partial \lambda_1}{\partial \psi_m} \\ \frac{\partial \lambda_2}{\partial \psi_m} \\ \frac{\partial \lambda_3}{\partial \psi_m} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ -C_{m\underline{z}} \\ -C_{m\underline{y}} \\ -C_{m\underline{x}} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For convenience this set of equations is written as

$$\underline{A}\ \underline{u} = \underline{w}$$

The solution in formal terms is

$$\underline{u} = \underline{A}^{-1}\underline{w}$$

Before continuing with this development, it is noted that the effect of a variation in ψ_m on the values of the cost measures is given by

$$\frac{\partial C_m}{\partial \psi_m} = \sum_{k=1}^{K} \frac{\partial C_m}{\partial x_k} \frac{\partial x_k}{\partial \psi_m} + \sum_{j=1}^{J} \frac{\partial C_m}{\partial y_j} \cdot \frac{\partial y_j}{\partial \psi_m} + \sum_{i=1}^{I} \frac{\partial C_m}{\partial z_i} \cdot \frac{\partial z_i}{\partial \psi_m}$$

$$(m = 1, \ldots, M)$$

Continuing with the main development again, the result of applying the procedure to the efficiency problem is the following set of equations.

$$\underline{\underline{B}} \begin{bmatrix} \dfrac{\partial \underline{z}}{\partial \psi_m} \\[2ex] \dfrac{\partial \underline{y}}{\partial \psi_m} \\[2ex] \dfrac{\partial \underline{x}}{\partial \psi_m} \\[2ex] \dfrac{\partial \mu_1}{\partial \psi_m} \\[2ex] \dfrac{\partial \mu_2}{\partial \psi_m} \\[2ex] \dfrac{\partial \mu_3}{\partial \psi_m} \end{bmatrix} = \begin{bmatrix} -C_{m\underline{z}} \\[2ex] -C_{m\underline{y}} \\[2ex] -C_{m\underline{x}} \\[2ex] 0 \\[2ex] 0 \\[2ex] 0 \end{bmatrix}$$

In more compact notation:

$$\underline{\underline{B}}\,\underline{r} = \underline{\beta}$$

with the formal solution $\underline{r} = \underline{\underline{B}}^{-1}\underline{\beta}$

Again, this equation set is seen to be a subset of the previous, and the original set may be written in partitioned form as

$$\begin{bmatrix} A_{11} & \underline{Q} \\ \underline{P} & \underline{B} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \underline{E}}{\partial \psi_m} \\ \underline{\delta} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{\beta} \end{bmatrix}$$

where

$$\underline{\delta} = \begin{bmatrix} \dfrac{\partial \underline{z}}{\partial \psi_m} \\ \dfrac{\partial \underline{y}}{\partial \psi_m} \\ \dfrac{\partial \underline{x}}{\partial \psi_m} \\ \dfrac{\partial \lambda_1}{\partial \psi_m} \\ \dfrac{\partial \lambda_2}{\partial \psi_m} \\ \dfrac{\partial \lambda_3}{\partial \psi_m} \end{bmatrix}$$

The solution is

$$\begin{bmatrix} \dfrac{\partial \underline{E}}{\partial \psi_m} \\ \underline{\delta} \end{bmatrix} = \begin{bmatrix} -(A_{11} - \underline{Q}\,\underline{B}^{-1}\,\underline{P})^{-1}\underline{Q}\,\underline{B}^{-1}\underline{\beta} \\ \underline{B}^{-1}\,\underline{\beta} + \underline{B}^{-1}\,\underline{P}(A_{11} - \underline{Q}\,\underline{B}^{-1}\,\underline{P})^{-1}\underline{Q}\,\underline{B}^{-1}\underline{\beta} \end{bmatrix}$$

so the solution for δ may be written as

$$\underline{\delta} = \underline{r} - \underline{B}^{-1}\,\underline{P}\,\frac{\partial \underline{E}}{\partial \psi_M}$$

Again, there is an efficiency substitution effect and a benefit effect.

When ϕ_ℓ is considered instead of ψ_m, the procedure is the same. The results are somewhat different in that ϕ_ℓ and ψ_m appear differently in equations (1) through (7). The results are as follows with $\underline{\delta}^*$ being the ϕ_ℓ version of $\underline{\delta}$.

$$\begin{bmatrix} \frac{\partial \underline{E}}{\partial \phi_\ell} \\ \underline{\delta}^* \end{bmatrix} = \begin{bmatrix} (A_{11}-\underline{Q}\underline{B}^{-1}\underline{P})^{-1} & -(\underline{A}_{11}-\underline{Q}\underline{B}^{-1}\underline{P})^{-1}\underline{Q}\underline{B}^{-1} \\ -\underline{B}^{-1}\underline{P}(A_{11}-\underline{Q}\underline{B}^{-1}\underline{P})^{-1} & \underline{B}^{-1}+\underline{B}^{-1}\underline{P}(\underline{A}-\underline{Q}\underline{B}^{-1}\underline{P})^{-1}\underline{Q}\underline{B}^{-1} \end{bmatrix} \begin{bmatrix} \underline{\epsilon}^{L\times 1} \\ \underline{0}^{I+J+K+3\times 1} \end{bmatrix}$$

where $\underline{\epsilon}$ is a vector of zero's except for a minus one in the ℓ^{th} component.

Explicitly, the solution for δ^* is

$$\underline{\delta}^* = -\underline{B}^{-1}\underline{P}\frac{\partial \underline{E}}{\partial \phi_\ell}$$

and there is only a benefit effect.

The comparative static analysis performed in the preceding paragraphs yields some interesting insights into the study team's problem. However, much further research is needed to determine the qualitative properties of the systems of equations.

THE IMPLICIT COST-BENEFIT FUNCTION

In many discussions of cost-benefit analysis an implicit form of the relationship of costs and benefits is used. For example, the following equation is given.

$$H(\underline{E},\underline{C}) = 0$$

It is the purpose of this part of the paper to discuss the relationship of such an implicit form to the previous disaggregated model.

The necessary conditions for the vector maximum problem of the study team (equations 1 - 7) can be solved for the choice variables as functions of the efficiency prices and the cost coefficients. That is, by use of the implicit function theorem applied to equations (1) through (7), the following equations can be developed.

$$\underline{x} = \underline{X}(\underline{\phi}, \underline{\psi}, \underline{r})$$

$$\underline{y} = \underline{Y}(\underline{\phi}, \underline{\psi}, \underline{r})$$

$$\underline{z} = \underline{Z}(\underline{\phi}, \underline{\psi}, \underline{r})$$

$$\underline{E} = \underline{E}(\underline{\phi}, \underline{\psi}, \underline{r})$$

$$\lambda_1 = \lambda_1(\underline{\phi}, \underline{\psi}, \underline{r})$$

$$\lambda_2 = \lambda_2(\underline{\phi}, \underline{\psi}, \underline{r})$$

$$\lambda_3 = \lambda_3(\underline{\phi}, \underline{\psi}, \underline{r})$$

In addition, it is known that

$$\underline{C} = \underline{C}(\underline{x}, \underline{y}, \underline{z}, \underline{r})$$

Substituting for $\underline{x}$, $\underline{y}$, $\underline{z}$, this equation becomes

$$\underline{C} = \underline{C}(\underline{X}(\underline{\phi}, \underline{\psi}, \underline{r}), \underline{Y}(\underline{\phi}, \underline{\psi}, \underline{r}), \underline{Z}(\underline{\phi}, \underline{\psi}, \underline{r}), \underline{r})$$

This set of equations, together with

$$\underline{E} = \underline{E}(\underline{\phi}, \underline{\psi}, \underline{r})$$

parametrically determine the cost-benefit surface.

These equations for $\underline{C}$ and $\underline{E}$ as functions of $\underline{\phi}$, $\underline{\psi}$ parametrically determine the cost-benefit surface (8, p. 371-375) since the sum of the ϕ's and ψ's is one. This latter theorem for vector maximum problems is proved by Karlin (6, p. 216-218). Hence, again using the implicit function theorem, $L + M - 1$ of the ϕ's and ψ's can be solved for as functions of the $L + M - 1$ C's and E's. In turn, these may be substituted into the remaining equation yielding, for example, the explicit form

$$E_L = f(E_1, \ldots, E_{L-1}, \underline{C})$$

which is easily transformed into implicit form. Hence, the vector maximum formulation permits the development of an implicit cost-benefit function.

Now that the cost-benefit surface is known in implicit function form, at least locally, it is of interest to study the qualitative properties of the surface. That is, the signs of output transformations, input substitutions, and marginal productivities are of interest. This line of research has not yet been pursued. It is noted that it involves a repeated application of the corollary to the implicit function theorem on the slopes of implicit functions (9).

SECTION IV

CONCLUSIONS

In this paper a mathematical model of an analysis team's results is presented. The variables are considered to have physical/social, time, space and state-of-nature attributes. The team's decision problem is formulated as a vector maximization problem. Decision rules are derived. Comparative statics results are given.

While a general framework for understanding the nature of a cost-benefit study is presented, it must be noted that this does not complete the research needed. There is opportunity for use of the Oualitative Calculus. There is opportunity to research the sociology of a team doing a study and the effect on the result. There is opportunity to show the nature of certain"simplifying" assumptions sometimes used.

REFERENCES

[1] Heuston, M. C. and Ogawa, G., "Observations on the Theoretical Basis of Cost-Effectiveness," Journal of the Operations Research Society of America, 14: 242-266 (1966).

[2] Miller, S. M., M. Rein, P. Roby, and B. M. Gross, "Poverty, Inequality and Conflict," The Annals of the American Academy of Political and Social Science, 373: 16-52 (1967).

[3] Debreu, G., Theory of Value, Wiley & Sons, New York (1958).

[4] Schlaifer, R., Analysis of Decisions Under Certainty; McGraw-Hill, New York (1967).

[5] Suppes, P. and J. L. Zinnes, "Basic Measurement Theory," Handbook of Mathematical Psychology, edited by R. D. Luce, R. Bush, E. Galanter, J. Wiley (1965).

[6] Karlin, S., Mathematical Methods and Theory in Games, Programming and Economics; Addison-Wesley, Reading, Massachusetts (1959).

[7] Kuhn, H. W. and A. W. Tucker, "Nonlinear Programming," Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, Jerzy Neyman, editor, University of California Press (1951).

[8] Taylor, A., Advanced Calculus; Ginn & Cox (1955).

[9] Hadley, G., Nonlinear and Dynamic Programming; Addison-Wesley, Reading, Massachusetts (1964).

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13. ABSTRACT

In this paper a mathematical model of an analysis team's results is presented. The variables are considered to have physical/social, time, space and state-of-nature attributes. The team's decision problem is formulated as a vector maximization problem.

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